Complex Variables

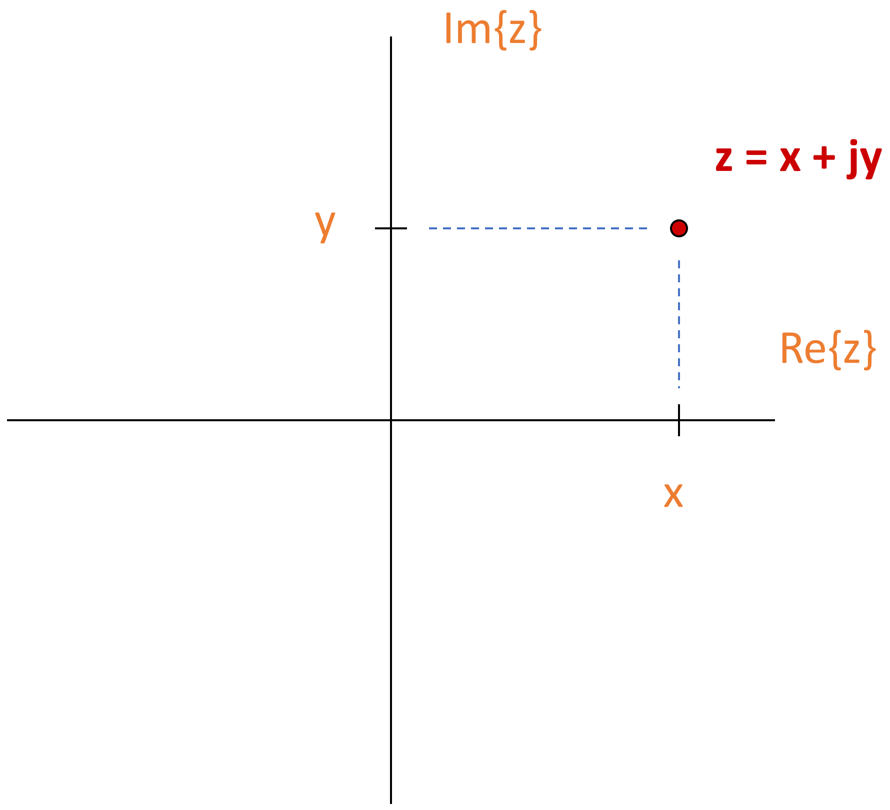
Complex numbers are really two numbers packaged into one entity (much like matrices). The two "numbers" are the real and imaginary portions of the complex number:

$$z = x + jy.$$

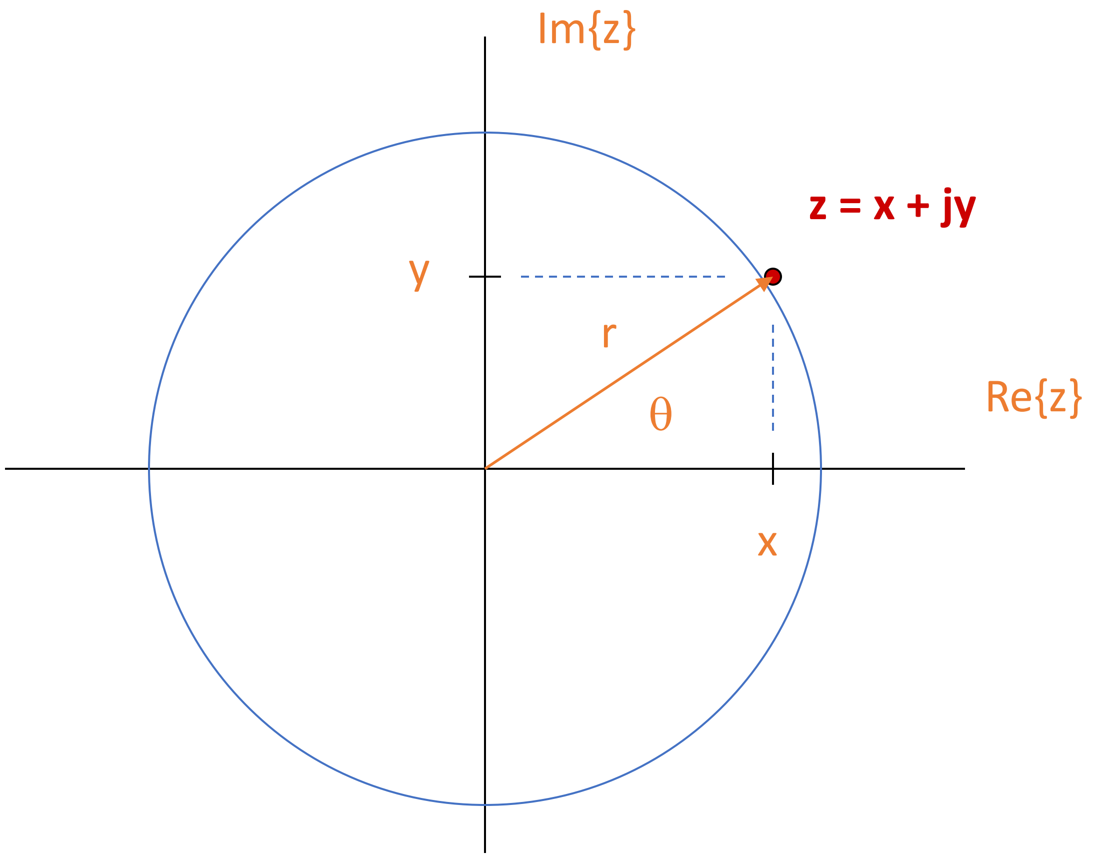
$$x = \mathrm{Re}\{z\}.$$

$$y = \mathrm{Im}\{z\}.$$

We may plot complex numbers in a ***complex plane***: the horizontal axis corresponds to the real part and the vertical axis corresponds to the imaginary part.



Often, we wish to use ***polar coordinates*** to specify the complex number.
Instead of horizontal ***x*** and vertical ***y***, we have radius ***r*** and angle θ.



The best way to express a complex number in polar coordinates is to use Euler's identity:

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

So,

$$z = re^{j\theta} = r\cos\theta + jr\sin\theta,$$

and

$$x = r\cos\theta.$$

$$y = r\sin\theta.$$

We also have

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2.$$

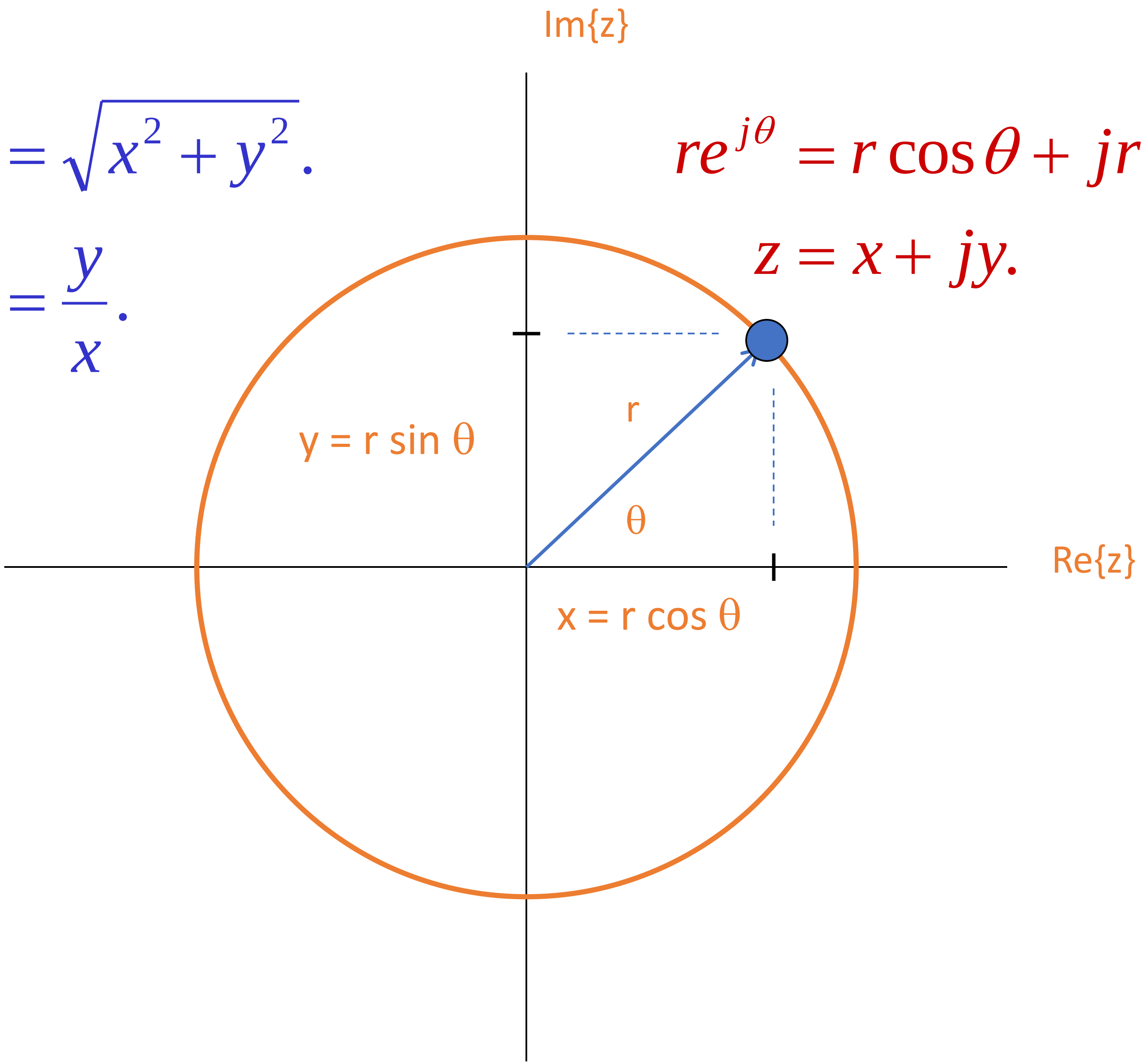
A summary of the complex relationships is on the following slide.

$$r = \sqrt{x^2 + y^2}.$$

$$\tan\theta = \frac{y}{x}.$$

$$re^{j\theta} = r\cos\theta + jr\sin\theta.$$

$$z = x + jy.$$



The ***magnitude*** of a complex number is the square-root of the sum of the squares of the real and imaginary parts:

$$|z| = |x + jy| = \sqrt{x^2 + y^2}.$$

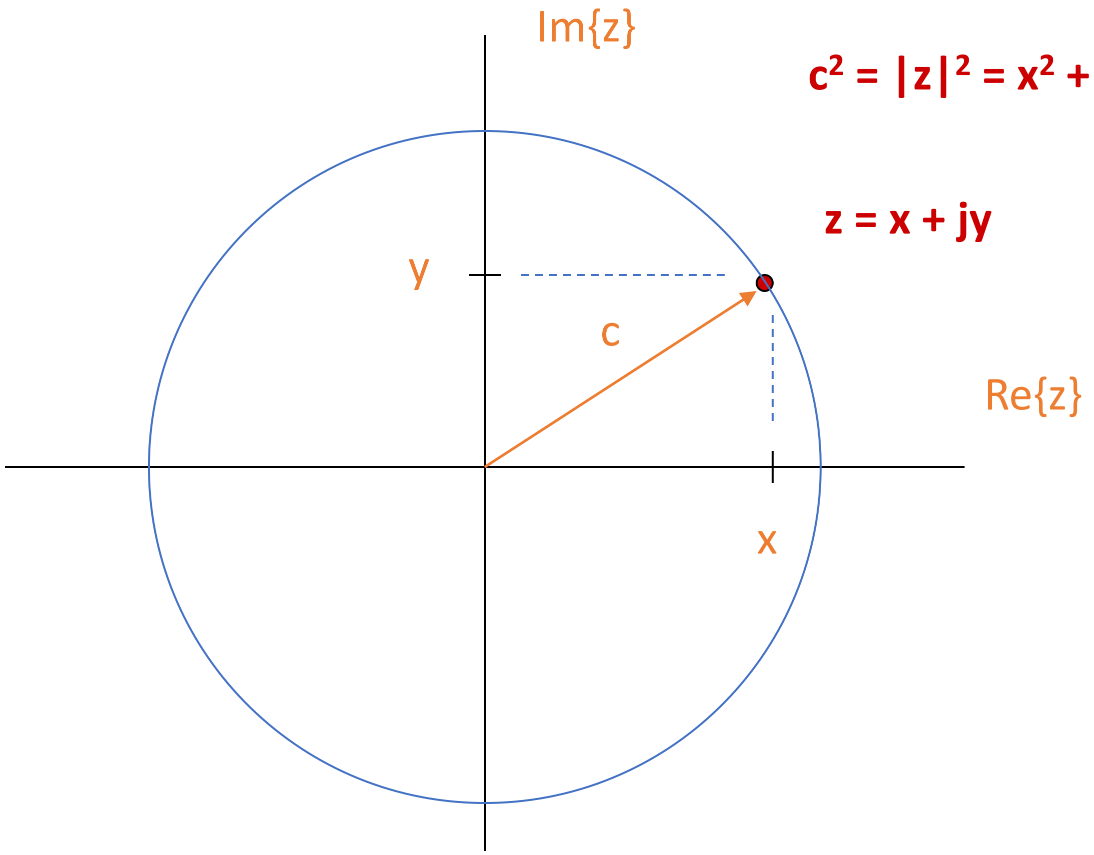
If we set the magnitude of a complex number equal to a *constant*, we have

$$|z| = |x + jy| = \sqrt{x^2 + y^2} = c,$$

or,

$$|z|^2 = x^2 + y^2 = c^2.$$

This is the equation of a *circle*, centered at the origin, of radius ***c***.



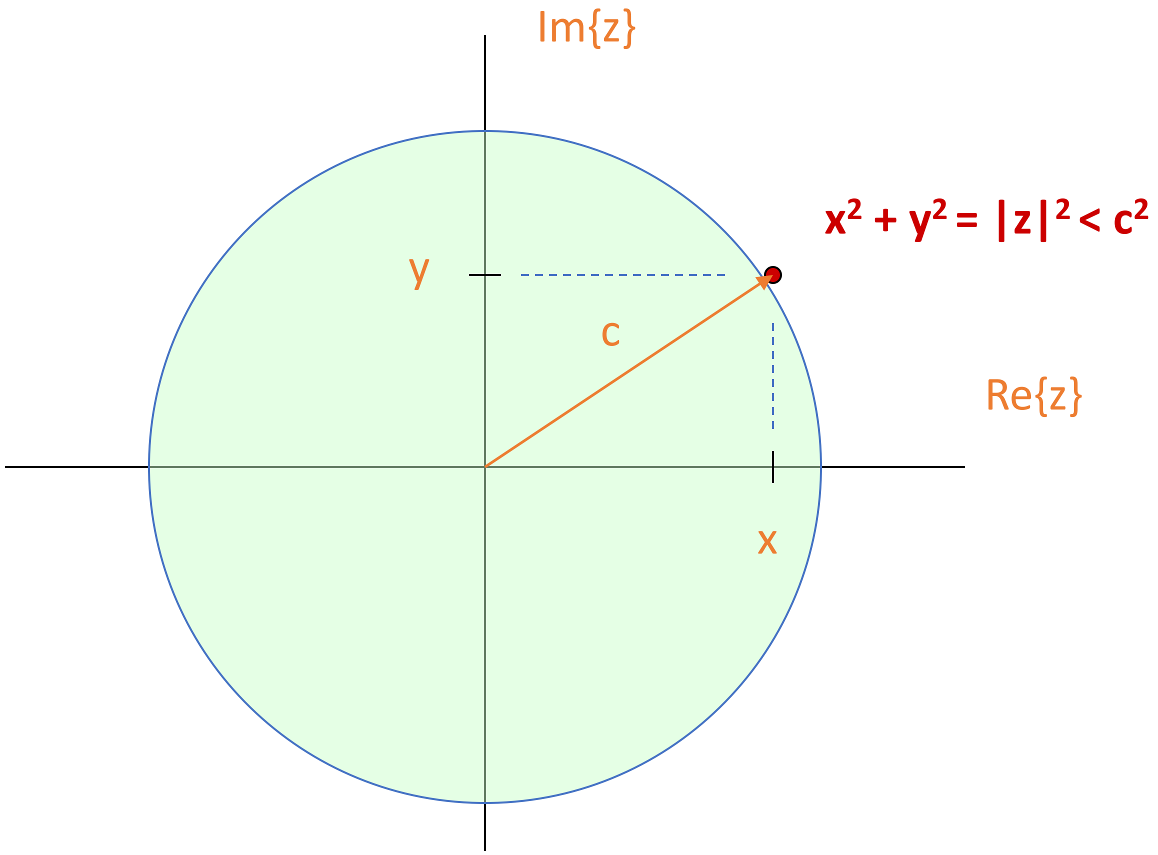
$c^2 = |z|^2 = x^2 + y^2$

$z = x + jy$

Suppose we wish to find the *region* corresponding to

$$|z|^2 = x^2 + y^2 < c^2.$$

This would be a *disk*, centered at the origin, of radius ***c***.


Im{z}
y
c
Re{z}
x
$x^2 + y^2 = |z|^2 < c^2$

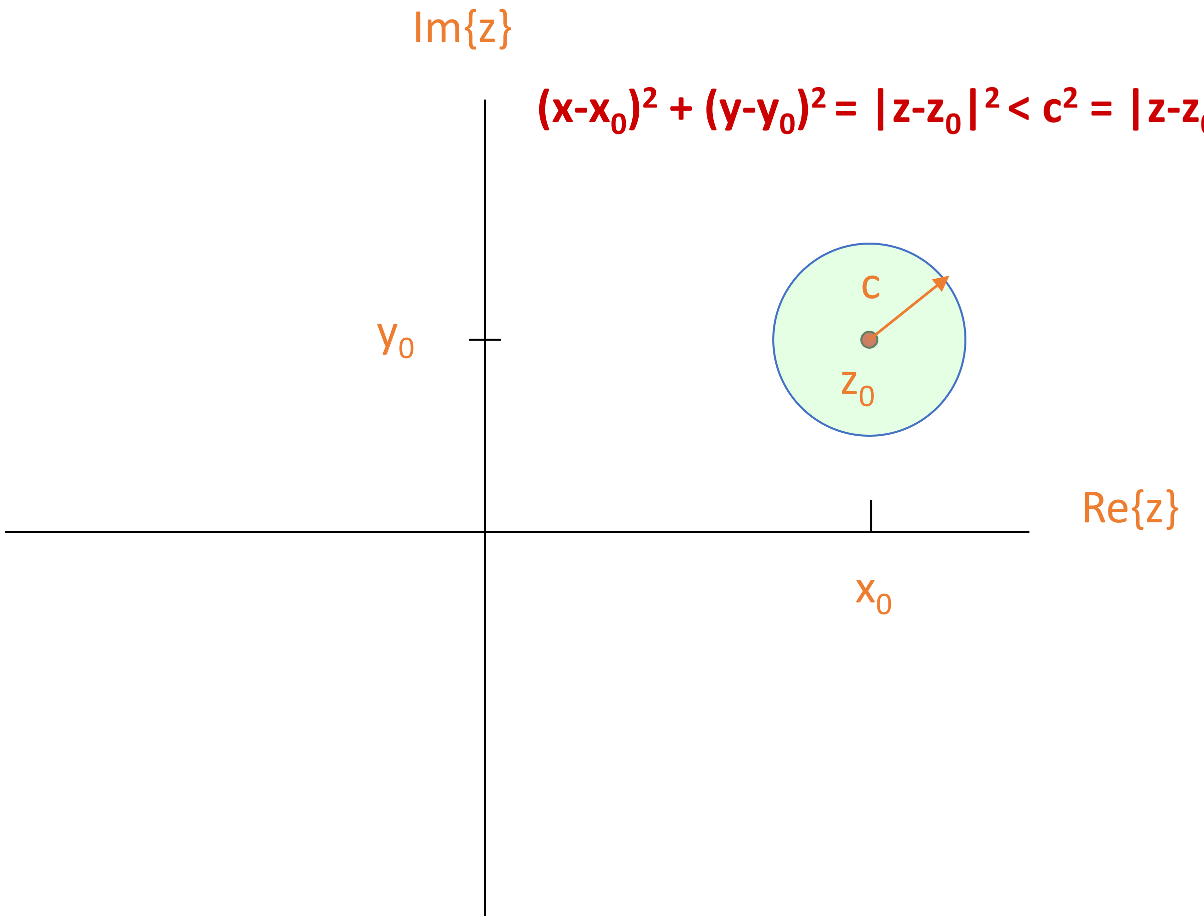
Suppose we wish to find the *region* corresponding to

$$|z - z_0|^2 < c^2.$$

This would be a *disk*, centered at z_0, of radius c.

$$|z - z_0|^2 = (x - x_0)^2 + (y - y_0)^2 < c^2.$$

$$(x-x_0)^2 + (y-y_0)^2 = |z-z_0|^2 < c^2 = |z-z_0|^2$$



Functions of Complex Variables

Suppose we had a function of a complex variable, say

$$w = f(z) = z^2.$$

Since ***z*** is a complex number, ***w*** will be a complex number. Since ***z*** has real and imaginary parts, ***w*** will have real and imaginary parts.

$$
\begin{aligned}
w = f(z) &= z^2 \\
&= (x + jy)^2 \\
&= x^2 + 2jxy - y^2 \\
&= \left[x^2 - y^2\right] + j\left[2xy\right].
\end{aligned}
$$

The standard notation for the real and imaginary parts of ***z*** are ***x*** and ***y*** respectively.

The standard notation for the real and imaginary parts of ***w*** are ***u*** and ***v*** respectively.

$$w = [x^2 - y^2] + j[2xy]$$
$$= u + jv,$$

where

$$u = [x^2 - y^2].$$
$$v = [2xy].$$

Both ***u*** and ***v*** are functions of ***x*** and ***y***.

So a complex function of one complex variable is really *two* real functions of *two* real variables.

$$w = f(z) = u(x, y) + jv(x, y).$$

Exercise: Find ***u(x,y)*** and ***v(x,y)*** for each of the following complex functions:

$$f(z) = z^3.$$

$$f(z) = e^z.$$

$$f(z) = e^{jz}.$$

$$f(z) = \cos z.$$

$$f(z) = \sqrt{z}.$$

$$f(z) = z^z.$$

Continuity of Complex Functions

In order to perform operations such as differentiation and integration of complex functions, we must be able to verify of the complex function is *continuous*.

A complex function

$$f(z)$$

is said to be *continuous at a point* z_0 if as z approaches z_0 (from any direction) then $f(z)$ can be made arbitrarily close to $f(z_0)$.

A more mathematical definition of continuity would be for any ε, we can make

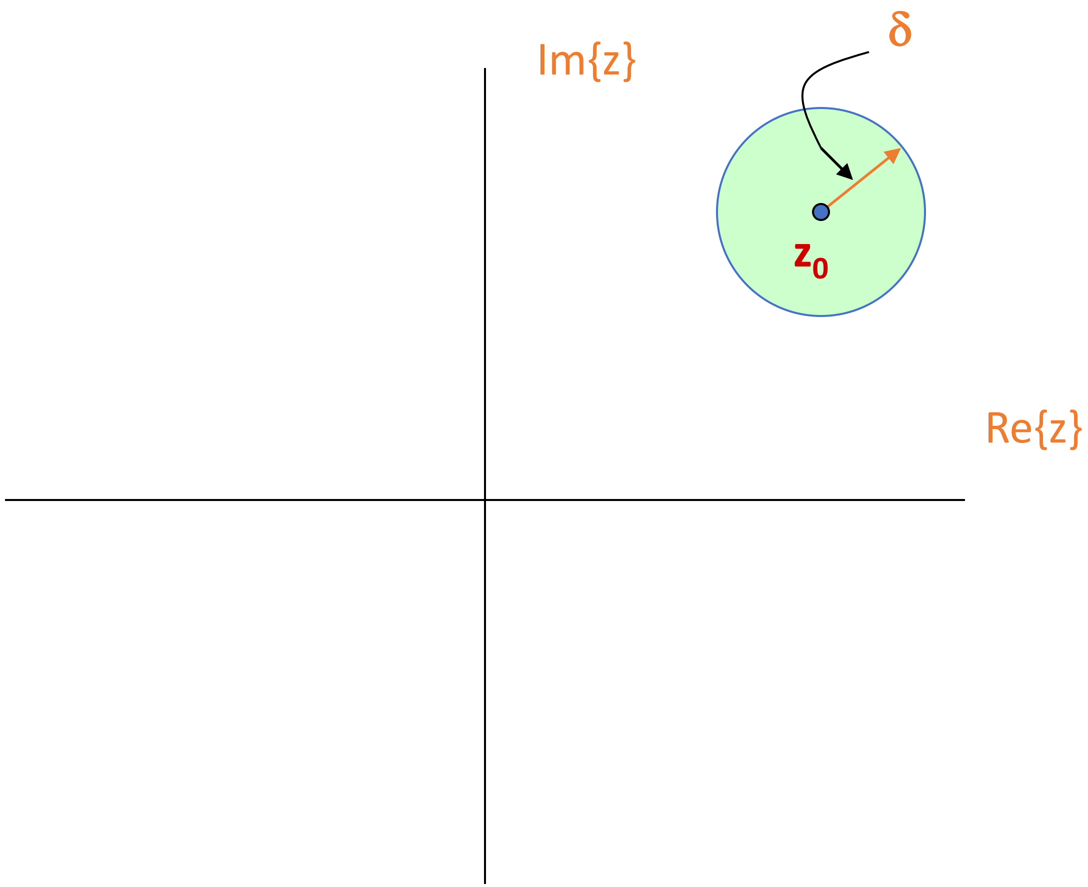
$$\left| f(z) - f(z_0) \right| < \varepsilon$$

for some δ such that

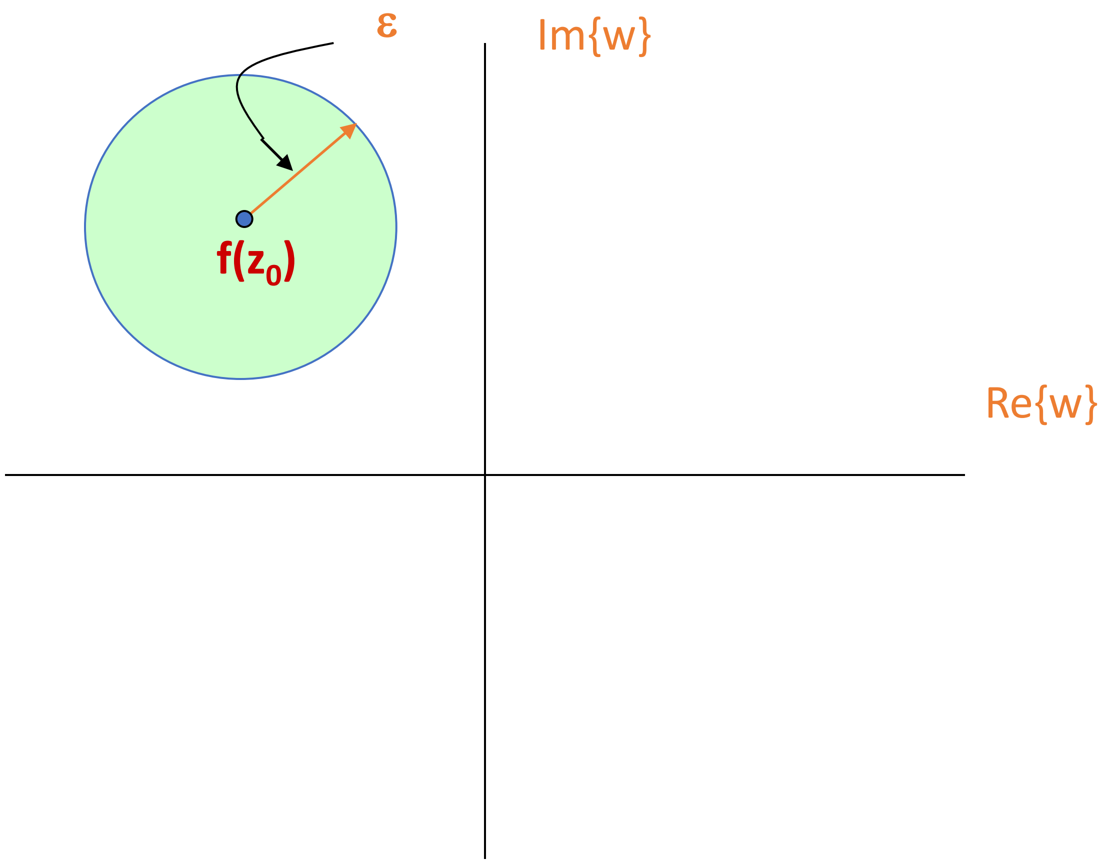
$$\left| z - z_0 \right| < \delta.$$

Since we are dealing with *complex numbers*, the *geometric* interpretation of this statement is different from that of real numbers.

The region $|z-z_0| < \delta$ defines a *disk* in the complex plane of radius δ centered about z_0.



So, if we wish $|f(z)-f(z_0)| < \varepsilon$ we must find a δ to make this so.



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